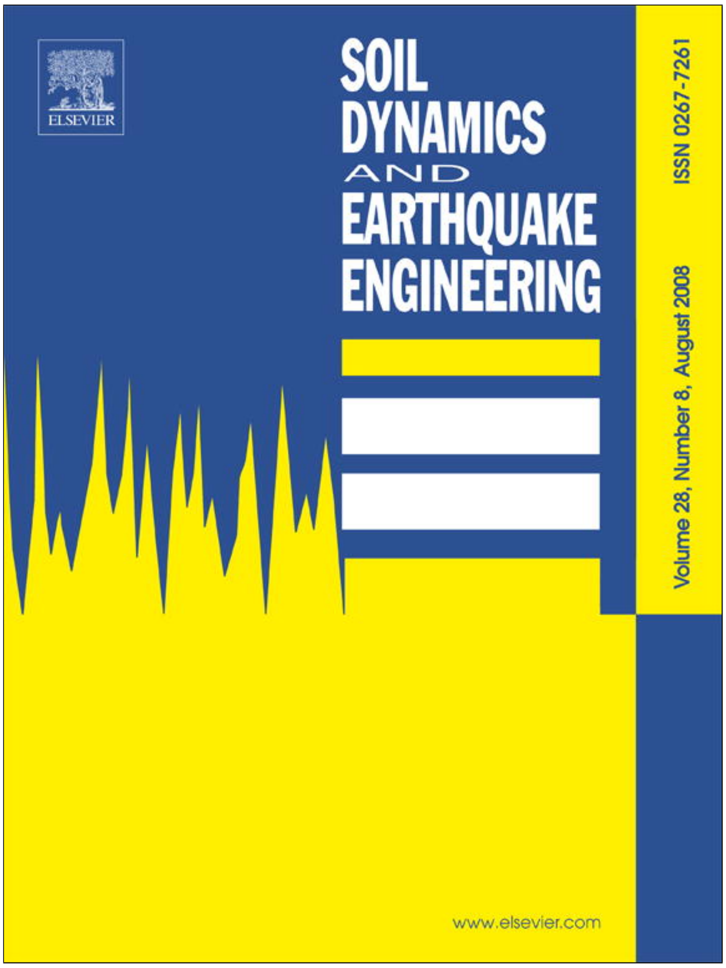
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# Response of pendulums to complex input ground motion

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## Abstract

Dynamic response of most seismological instruments and many engineering structures to ground shaking can be represented via response of a pendulum (single-degree-of-freedom oscillator). In most studies, pendulum response is simplified by considering the input from uni-axial translational motion alone. Complete ground motion however, includes not only translational components but also rotations (tilt and torsion). In this paper, complete equations of motion for three following types of pendulum are described: (i) conventional (mass-on-rod), (ii) mass-on-spring type, and (iii) inverted (astatic), then their response sensitivities to each component of complex ground motion are examined. The results of this study show that a horizontal pendulum similar to an accelerometer used in strong motion measurements is practically sensitive to translational motion and tilt only, while inverted pendulum commonly utilized to idealize multi-degree-of-freedom systems is sensitive not only to translational components, but also to angular accelerations and tilt. For better understanding of the inverted pendulum's dynamic behavior under complex ground excitation, relative contribution of each component of motion on response variants is carefully isolated. The systematically applied loading protocols indicate that vertical component of motion may create time-dependent variations on pendulum's oscillation period; yet most dramatic impact on response is produced by the tilting (rocking) component.




*Keywords:* Pendulum; Single-degree-of-freedom; Rotation; Tilt; Bridge pier; Seismometry

## 1. Introduction

Movement of the ground produced by an earthquake is a combination of translational and rotational motions. Many types of seismic waves propagating in the media create both translational and rotational excitations. For example, propagating Rayleigh waves can create significant rocking (tilting), and Love waves can create torsional excitations of the ground surface (e.g., Trifunac [1]). In mean time, traditional seismology and engineering practice consider only translational motions. Since 1970s, a number of attempts were made to measure or estimate rotational component of strong ground motion, but still there are no consistent measurements of rotations together with translational motion. According to theoretical and recent experimental data, rotational motions generated by an earthquake source can be significant in the near-fault area and reach up to $10^{-4}$ rad (e.g., [2–4]). Higher amplitude tilts in the order of up to$10^{-2}$ rad resulting from local site effects induced by strong ground shaking were also reported [5,6].

Conventional seismometer (velocimeter or accelerometer) records ground shaking during an earthquake via its classical pendulum setup composed of an inertial mass usually suspended in a frame by a combination of a spring, and a damper. This setup is aimed to prevent long-term oscillations in response to shaking, thereby acts as a physical low-cut filter. Similar to pendulums used in a typical seismometer, a number of engineering structures' response to earthquakes can be represented by the response of a SDOF oscillator (i.e., pendulum). Classically, response of an elastic SDOF oscillator in a seismometer is described by the ordinary differential equation of the second order:

$$y'' + 2\omega_n D_n y' + \omega_n^2 y = -x'', \qquad (1)$$

where $y$ is the recorded response of the instrument, $l_n$ is the length of pendulum arm, $\theta$ is the angle of pendulum rotation from the equilibrium, $y = \theta l_n$ for small angles of $\theta$,

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$\omega_n$ and $D_n$ are, respectively, the natural circular frequency and fraction of critical damping of the oscillator, $x''$ is the ground motion acceleration.

Close-form solutions of Eq. (1) in time-domain for various damping ratios are given by the Duhamel's integral [7]:

$$\theta(t) = \frac{1}{l_n v_n}\int_0^t \exp[-\omega_n D_n(t-\tau)]x''(\tau) \times \sin[v_n(t-\tau)]\mathrm{d}\tau, \quad D_n < 1,$$
$$\theta(t) = \frac{1}{l_n}\int_0^t \exp[-\omega_n D_n(t-\tau)]x''(\tau)\mathrm{d}\tau, \quad D_n = 1,$$
$$\theta(t) = \frac{1}{l_n \bar{v}_n}\int_0^t \exp[-\omega_n D_n(t-\tau)]x''(\tau) \times sh[\bar{v}_n(t-\tau)]\mathrm{d}\tau, \quad D_n > 1, \tag{2}$$

where $v_n = \omega_n\sqrt{1-D_n^2}$ for $D_n<1$, and $\bar{v}_n = \omega_n\sqrt{D_n^2-1}$ for $D_n>1$. Parameter $v_n$ is also called damped natural frequency since it is the frequency at which under-damped SDOF system oscillates freely. Most instruments (accelerometers and seismometers) used in seismology have damping ratio about 0.6–0.7, whereas many structural systems' damping ratio remains lower than 0.1. Typical velocimeters have damping ratio of $D_n \gg 1$. The solutions given via Eq. (2) are valid for small angles ($\theta$) of pendulum's relative rotation assuming that there are no rotations and tilts of the pendulum base.

### 1.1. Types of pendulum

According to terminology commonly used in seismology and followed in this paper, a horizontal pendulum is a pendulum sensitive to the horizontal input ground motion, and a vertical pendulum is sensitive to the motion along the vertical axis. In this study, the following three types of horizontal pendulums are considered:

1. Mass-on-rod type pendulum: oscillating in a horizontal plane with rotation axis being vertical (most seismic instruments are of this type) (Fig. 1).
2. Mass-on-spring type pendulum: oscillating in a horizontal plane (Fig. 2).
3. Inverted (astatic) mass-on-rod type pendulum: oscillating in a vertical plane around the horizontal axis (Fig. 3). Classical Wiechert's horizontal seismograph built around 1905 and still used at some seismological observatories represents an inverted pendulum. Numerous engineering structures including buildings, elevated water tanks, tower cranes, communication towers, poles etc. can also be idealized as a pendulum of this type.

The responses of vertical pendulums of the first two types are also considered in the course of this study. Note that mass-on-rod type vertical pendulum is oscillating around a horizontal axis of rotation. The theory of pendulums described in this paper is applicable for small

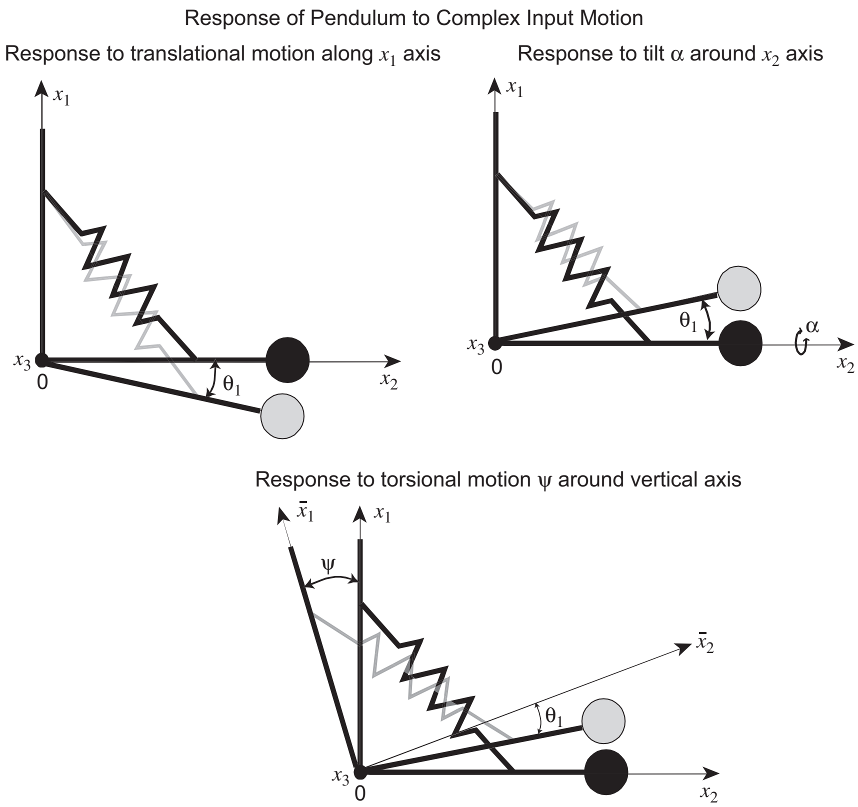


Fig. 1. Mass-on-rod pendulum and its response to complex ground motion (*note*: $x_1$ and $x_2$ lie in horizontal plane).

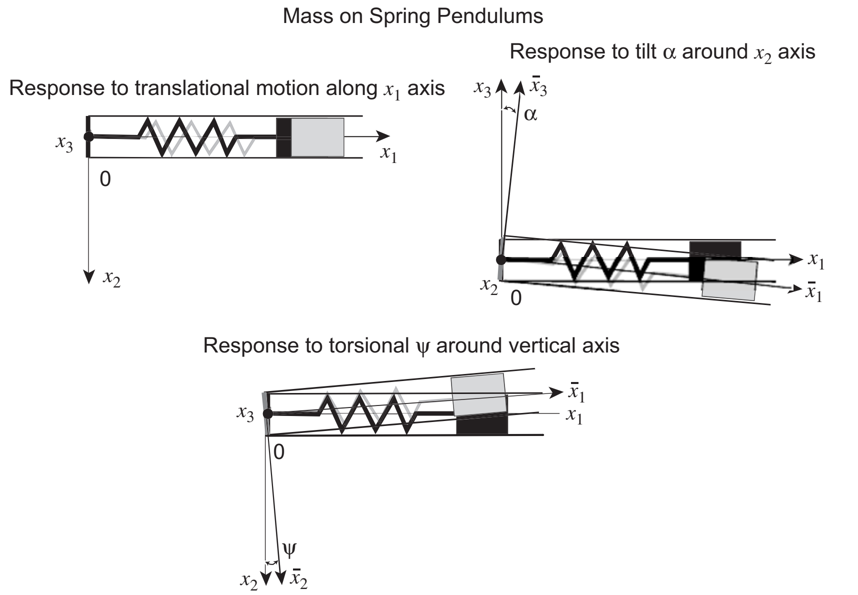


Fig. 2. Mass-on-spring pendulum and its response to complex ground motion (*note*: $x_1$ and $x_2$ lie in horizontal plane).

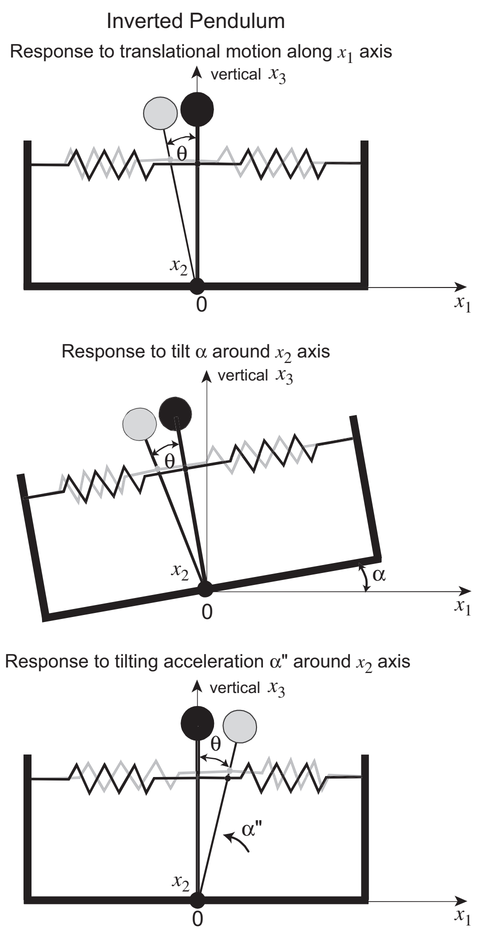


Fig. 3. Inverted (astatic) pendulum and its response to complex ground motion (*note*: $x_1$ and $x_2$ lie in horizontal plane).

oscillations assuming that pendulum's declination from the equilibrium does not exceed few degrees. As a first approximation, Eq. (1) is sufficient to describe the behavior of all above-mentioned types of pendulum including vertical and horizontal.

### 1.2. *Configuration of sensors in seismological instruments*

Standard configuration of sensors in seismological measurements includes three instruments: two sensors are oriented horizontally perpendicular to each other and the third sensor is oriented vertically representing Cartesian coordinate system. The second configuration commonly used for petroleum exploration is so-called Galperin's or symmetric configuration [8] developed as a tool for three-component borehole studies. In this configuration, the three sensors are also positioned orthogonally with respect to each other, but all three sensors are tilted at the same angle to the vertical axis (triaxial system of coordinate balanced on its corner). This configuration ensures that each of the three identical, single-component sensors respond equally to gravity. Sensors are mounted at an angle of 35.3° to the horizontal in vertical plane (54.7° to the vertical axis) and at 120° relative to each other in horizontal plane (azimuths between sensors are 120° relative to each other. Advantage of this configuration is that all three sensors could be identical in design. Records obtained using Galperin's configuration must be rotated into an earth referenced $X$, $Y$, $Z$ coordinate system. The disadvantage of this configuration is that if one of the sensors is not working properly, it results in degradation of all three components measured.

Most seismological instruments are using standard (East, North, Up) configuration (e.g. Streckeisen STS-1 and Guralp CMG-3 T). But some seismometers (e.g. Streckeisen STS-2, Trillium of Nanometrics and Cronos of Kinemetrics) adopted Galperin's configuration. In this paper, classical configuration is considered solely due to its more common applications.

## 2. Complete equation of small motion of pendulums

### 2.1. Complete equations of conventional pendulum motion

Most of the seismological sensors (seismometers and accelerometers) used in conventional seismological instruments are pendulums of the mass-on-rod type (Fig. 1). Accordingly, complete equation of small oscillations (i.e. $\sin\theta \cong \theta$) of the horizontal pendulum shown in Fig. 1 can be expressed as

$$y_1'' + 2\omega_1 D_1 y_1' + \omega_1^2 y_1 = -x_1'' + g\alpha - l_1\psi'' + x_2''\theta_1, \tag{3}$$

where $g$ is acceleration due to gravity, $\alpha$ is tilting of the ground, and $\psi$ is torsion (rotation around vertical axis). The fourth term in Eq. (3) is usually called cross-axis sensitivity (when pendulum is out of equilibrium, it is also sensitive to the acceleration along the direction perpendicular to the axis of sensitivity). Transferring the last term in the right-hand side related to the angle of rotation of pendulum (i.e. $x_2''\theta_1 =$ output of the pendulum) to the left-hand side of Eq. (3) yields:

$$y_1'' + 2\omega_1 D_1 y_1' + (\omega_1^2 - x_2''/l_1)y_1 = -x_1'' + g\alpha - l_1\psi''. \tag{4}$$

It should be noted that a horizontal pendulum sensitive to translational acceleration is also sensitive to tilt $g\alpha$, angular acceleration $l_n\psi''$, and acceleration along perpendicular direction $x_2''\theta_1$ [9,10].

Sensitivity of the vertical pendulum to tilts is proportional to cos($\alpha$), thereby its contribution can be neglected for small tilting angles. The resultant equation of motion of a vertical pendulum can be written as

$$y_3'' + 2\omega_3 D_3 y_3' + \omega_3^2 y_3 = -z'' - l_3\alpha'' + x_2''\theta_3, \tag{5}$$

where $z''$ is acceleration in the vertical direction. Transferring cross-axis sensitivity to the left-hand side of Eq. (5) results in

$$y_3'' + 2\omega_3 D_3 y_3' + (\omega_3^2 - x_2''/l_3)y_3 = -z'' - l_3\alpha''. \tag{6}$$

Note that a vertical pendulum is sensitive to vertical motion and should not be confused with an inverted pendulum that is sensitive to horizontal motion. While a horizontal pendulum (Eq. (3)) is sensitive to the acceleration of linear motion, tilt, angular acceleration, and cross-axis excitations, a vertical pendulum is sensitive to the acceleration of linear motion, angular acceleration, and cross-axis excitations (Eq. (5)). Unfortunately, the completeness of representing equation of pendulum motion in the seismological literature varies. For instance, Golitsyn [11] does not take into account the cross-axis sensitivity, while Aki and Richards [12] following Rodgers [13] do not consider the angular acceleration term. Response of a vertical pendulum and a horizontal pendulum oscillating in a horizontal plane is considered in a number of publications (e.g. [9,14–17]). These studies show that the last terms in the right side of Eqs. (3) and (5) (cross-axis sensitivity) is relatively small and can be neglected in most applications.

In general, Eqs. (5) and (6) are non-linear, since the third term in the left side of Eq. (5) is response parameter thus it is time and amplitude dependent. In contrast to the linear differential Eq. (1), the non-linear differential form of Eqs. (4)–(6) do not have direct solutions. But for $D_1 \ll 1$ and for $\omega_1^2 \gg x_2''(t)/l_1$ their solutions (similarly to the solution (2) of Eq. (1)) can be approximated as

$$\theta_1(t) \approx \frac{1}{l_1 v_{\mathrm{eff}}(t)} \int_0^t \exp[-\omega_{\mathrm{eff}}(\tau) D_1 (t-\tau)] \times F(\tau) \sin[v_{\mathrm{eff}}(\tau)(t-\tau)]\mathrm{d}\tau, \tag{7}$$

where

$$F(t) = -x_1''(t) + g\alpha(t) - l_1\psi''(t),$$
$$\omega_{\mathrm{eff}}(t) = \sqrt{\omega_1^2 - x_2''(t)/l_1},$$
$$v_{\mathrm{eff}}(t) = \omega_{\mathrm{eff}}(t) \cdot \sqrt{1 - D_1^2}. \tag{8}$$

$F(t)$ is a complex input forcing function and $\omega_{\mathrm{eff}}(t)$ is effective time dependent frequency function. Eqs. (7) and (8) represent solutions for the horizontal sensor.

Solution of Eq. (6) for the vertical sensor is the same as for the horizontal one, except for its forcing function:

$$F(t) = -z''(t) - l_3\alpha''(t). \tag{9}$$

It is important to underscore that those equations represent elastic response of pendulums (as material behavior), with non-linearity created by time and amplitude dependence of equation coefficient.

### 2.2. Complete equations of mass-on-spring pendulum motion

Schematic representation of a mass-on-spring type seismometer is shown in Fig. 2. It is assumed that mass oscillates without friction. This type of pendulum is commonly used in geophysical prospecting (geophone), but it is also used in some accelerometers (e.g. SMACH accelerograph made in Switzerland by GeoSIG). Complete equation of this pendulum is similar to Eq. (3) except it is not sensitive to torsional acceleration, and cross-axis sensitivity is proportional to tilt of the base [9]:

$$y_1'' + 2\omega_1 D_1 y_1' + \omega_1^2 y_1 = -x_1'' + g\alpha + x_2''\alpha. \tag{10}$$

Theoretically, centrifugal acceleration $a_c = l_n(\psi')^2$. also contributes to the oscillations of the pendulum. Yet, it is realistically of lower order and can be neglected, since pendulum arm of this type of seismometer usually does not exceed 20 cm and the highest angular velocity observed during earthquakes does not exceed 0.2 rad/s [6].

Sensitivity of the vertical pendulum of the mass-on-spring type to tilts is proportional to cos($\alpha$) and correspondingly can be neglected for small tilt angles [9]. Consequently, equation of motions for a vertical sensor can be simplified to

$$y_3'' + 2\omega_3 D_3 y_3' + \omega_3^2 y_3 = -z'' + x_2''\alpha. \tag{11}$$

Since cross-axis sensitivity term is proportional to tilt of pendulum base (input signal), Eqs. (10) and (11) remain linear. Realistically, mass-on-spring pendulums can also demonstrate non-linearity as a result of non-constant damping due to friction.

### 2.3. Complete equations of inverted pendulum motion

In this study, a special attention is devoted to the inverted (astatic) pendulum (Fig. 3). Its response is sensitive to the horizontal ground motion with horizontal axis of rotation. Although this type of pendulum has limited use in seismology, it has significant importance for engineering applications where it is often used to simulate the dynamic response of various structural systems. In case of a complex input ground motion that includes tilting of the base of pendulum ($\alpha$), the complete equation of motion takes the following form [18]:

$$y_1'' + 2\omega_1 D_1 y_1' + \omega_1^2 y_1 = -x_1'' + (g+z'')\alpha + (g+z'')\theta_1 - l_1\alpha'', \tag{12}$$

where $\alpha''$ is angular acceleration of tilt, and $g$ is acceleration due to gravity.

Note that input force in the right-hand side of Eq. (12) has more entities than Eq. (4) of the conventional (mass-on-rod) pendulum. An inverted pendulum is sensitive to horizontal acceleration along the $x$-axis (first term in the right-hand side of Eq. (12)), tilt of the base (second term), additional tilt of pendulum due to gravity and vertical acceleration (third term), and angular acceleration of tilting (fourth term). Eq. (12) is an inclusive equation showing response of an inverted pendulum to a complex ground motion including translational and rotational components. Moving the entities related to the relative rotation of pendulum to the left-hand side reads the following expression:

$$y_1'' + 2\omega_1 D_1 y_1' + [\omega_1^2 - (g+z'')/l_1]y_1 = -x_1'' + (g+z'')\alpha - l_1\alpha''. \tag{13}$$

Eq. (13) of an inverted pendulum and Eq. (4) of a conventional pendulum used in seismometers demonstrate the following differences:

1. The right-hand side of Eq. (13) and the response of an inverted pendulum are dependent upon different set of input functions than that of Eq. (4).
2. The non-linearity of Eq. (13) is much higher than that of Eq. (4) because of gravity factor influencing the effective frequency of oscillation (third term in the left-hand side of Eq. (13)).
3. As shown in a number of studies [9,14–17] cross-axis sensitivity can affect response of a conventional accelerometer only in conditions when amplitude of motions in the direction $x_2$, perpendicular to the main sensitivity along the $x_1$, direction is significantly higher than 1.0$g$. Consequently, non-linearity in Eq. (4) only appears when cross-axis amplitudes are extremely high. Because of sensitivity of inverted pendulum to gravity, Eq. (13) becomes linear only if $\omega_1^2 \gg [g(t)+z''(t)]/l_1$. The higher is the natural frequency of a pendulum the lower is the non-linearity of Eq. (13). Non-linearity effect diminishes for pendulums with longer arm assuming that natural frequency remains unchanged.

Similar to Eq. (4), differential Eq. (13) is non-linear and does not have direct solution. But for $D_1 \ll 1$ and for $\omega_1^2 \gg [g(t)+z''(t)]/l_1$, its solution can be approximated by using approach as given in Eq. (7):

$$\theta_1(t) \approx \frac{1}{l_1 v_{\mathrm{eff}}(t)} \int_0^t \exp[-\omega_{\mathrm{eff}}(\tau) D_1 (t-\tau)] \times F(\tau)\sin[v_{\mathrm{eff}}(\tau)(t-\tau)]\mathrm{d}\tau, \tag{14}$$

but with different forcing and effective frequency functions:

$$F(t) = -x_1''(t) + [g+z''(t)]\alpha(t) - l_1\alpha''(t),$$
$$\omega_{\mathrm{eff}}(t) = \sqrt{\omega_1^2 - [g(t)+z''(t)]/l_1},$$
$$v_{\mathrm{eff}}(t) = \omega_{\mathrm{eff}}(t)\sqrt{1-D_1^2}. \tag{15}$$

Comparing solution (Eq. (14)) of Eq. (13) with solution (Eq. (2)) of Eq. (1), one can detect two major differences: (i) input motion becomes much more complex and constant parameter natural frequency $\omega_n$ is replaced by the time-variant oscillation frequency $\omega_{\mathrm{eff}}(t)$. Eq. (14) is more complicated to study than Eq. (1) since it has added complexity in the excitation function $F(t)$ including vertical acceleration, tilt and angular accelerations of tilt.

Effective frequency $\omega_{\mathrm{eff}}(t)$ replaces the constant parameter $\omega_1$ (natural frequency of pendulum). Right-hand side of Eq. (15) is positive and real when

$$\omega_1^2 > [g(t)+z''(t)]/l_1.$$

Investigation of pendulum's response to the complex ground motion becomes important in application to the strong ground shaking that generally happens in the near field of an earthquake source. In the preceding sections, the influence of different forcing functions in the right-hand side of Eq. (13) on the outcomes of pendulum is systematically investigated. Particular attention is devoted to the coupled effects of vertical acceleration and gravity on response (variability of one of the coefficients in the left-hand side of Eq. (13)).

## 3. Testing procedure for relative impacts of ground motion components

As an input for testing the pendulums response, we use corrected acceleration (Fig. 4) and tilt (Fig. 5) obtained from the record of the Mw 6.7 Northridge earthquake of 1994 recorded at Pacoima Dam Upper left abutment station by the California Strong Motion Instrumentation Program [19,20]. Approximate method used to extract tilts

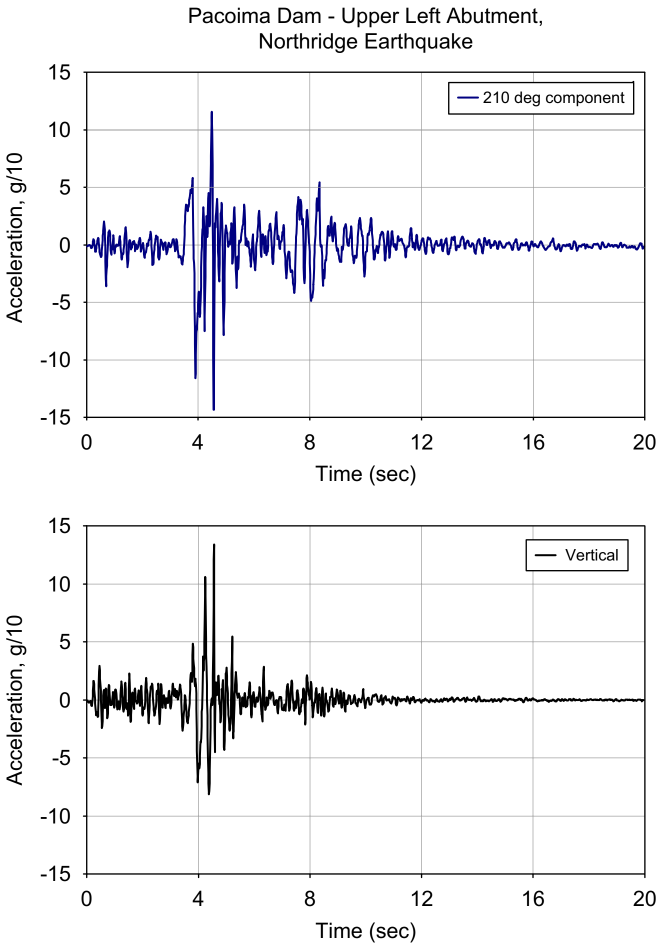


Fig. 4. Recorded horizontal and vertical accelerations at the Pacoima dam—upper left abutment during the 1994 Northridge earthquake.

from the three-component recorded accelerogram is based on a difference in tilt sensitivity of vertical and horizontal sensors and is described with details by Graizer [6].

### 3.1. Modeling accelerometer response

In the first series of numerical tests, we studied the complex response of an accelerometer similar to one used in SMA-1 (Fig. 6). Standard parameters of these sensors are: natural frequency of about 25 Hz ($T_n = 0.04$ s), damping ratio of $D_n = 0.65$ and pendulum arm of $l_n \sim 1.5$ cm.

In order to study the effect for each of ground motion components separately, the following computations and comparisons were conducted:

1. The response of an oscillator to purely translational motion equivalent to the solution of Eq. (1) was computed.
2. The effect of cross-axis sensitivity $x_2''\theta_1$ was considered by taking into account time variations of the natural frequency (Eqs. (7) and (8)).
3. Influence of angular acceleration $l_n\psi''$ was considered.
4. Effect of tilt $g\alpha$ was included.

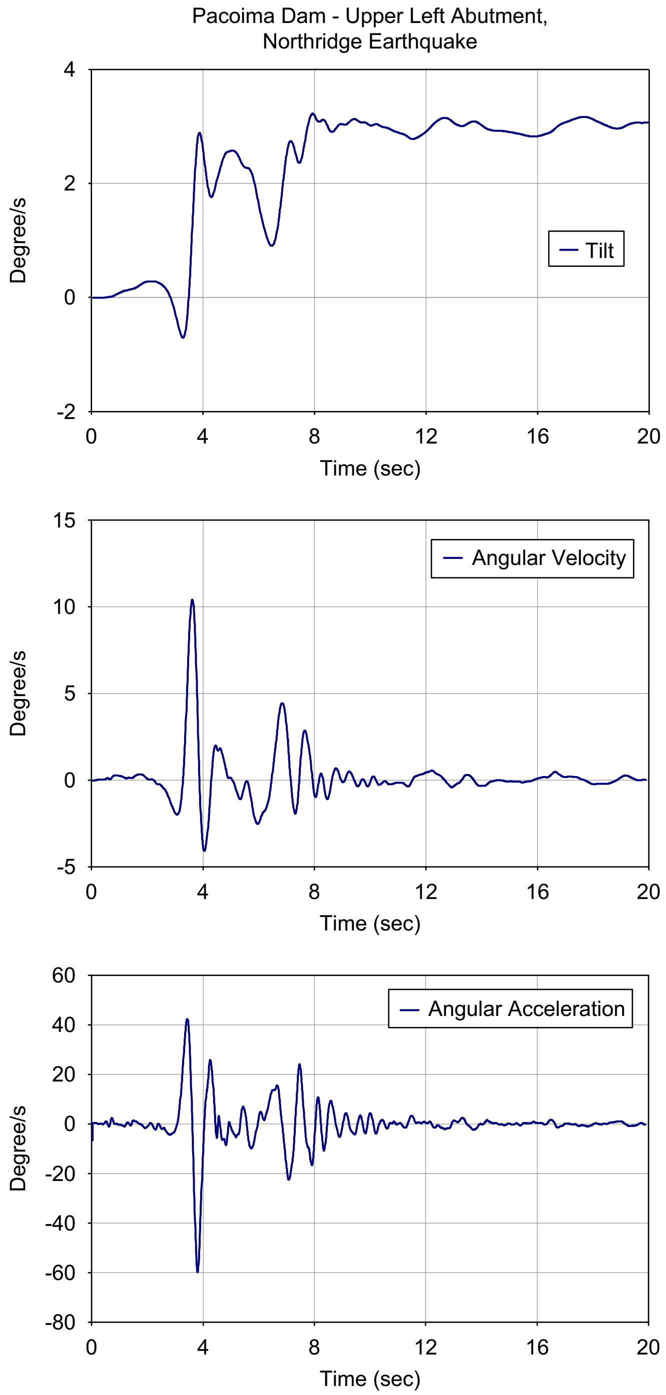


Fig. 5. Tilt, tilting velocity and tilting acceleration at the Pacoima dam—upper left abutment during the 1994 Northridge earthquake.

The numerical tests confirmed the results previously shown in Graizer [9,14] that

1. Cross-axis sensitivity is practically very low even for cross-axis acceleration of more than 1.0$g$. The difference in amplitudes between purely translational response and that of accounting for cross-axis input is less than 0.1%. This effect is so low because $\omega_1^2 \gg x_2''(t)/l_1$ for an accelerometer with 25 Hz natural frequency. To make

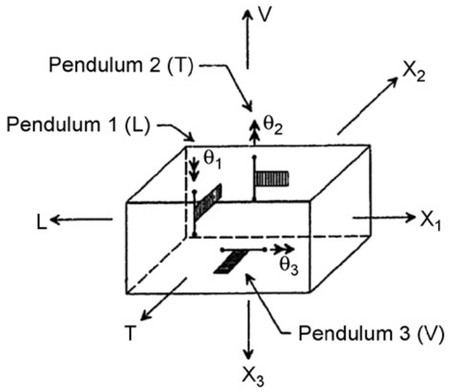


Fig. 6. Schematic representation of three transducers in an accelerograph. The coordinate axes $X_1$, $X_2$ and $X_3$ serve to describe the motion of the $L$, $T$ and $V$ transducers, respectively. Angles $\theta_1$, $\theta_2$ and $\theta_3$ describe the deflection of the transducer pendulums (modified from Trifunac and Todorovska [17]).

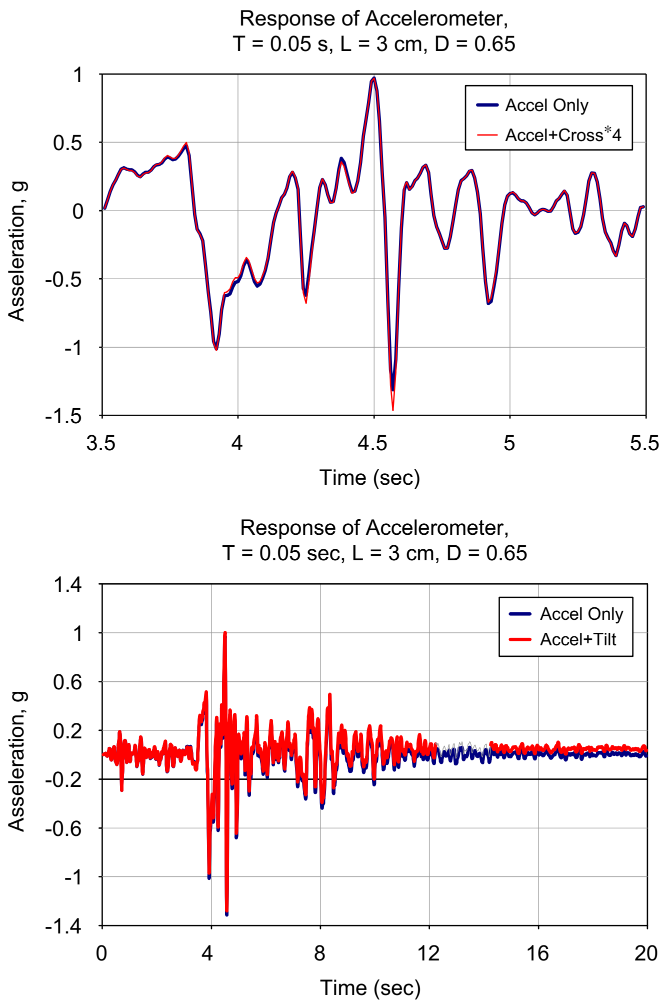


Fig. 7. Response of an accelerometer to cross-axis acceleration (top-panel) and tilt (bottom-panel).

the effect of cross-axis sensitivity visible, acceleration along the perpendicular direction is significantly amplified. Fig. 7(top) demonstrates the effect of cross-axis sensitivity for the exaggerated intensities, when acceleration along the perpendicular axis was artificially multiplied four times reaching maximum amplitude of more than $5.0g$. Even in such extreme case, the difference is only visible in a few places without exceeding 10%.

2. Effect of angular acceleration on an accelerometer response is extremely low, since the length of pendulum arm is very short. Even for relatively very high angular accelerations of up to $60^\circ/s^2$ ($\sim 1\ rad/s^2$) this effect is still invisible.
3. Tilt produces significant influence on response of pendulum. Residual tilt resulting in a shift of final position of pendulum (Fig. 7(bottom)) can be visually recognized. This effect looks like a baseline shift. As shown in Graizer [10] dynamic tilt introduces erroneous long-period noise and makes calculations of residual displacement impossible.

Modern accelerographs use sensors with standard natural frequency of 50 Hz and higher. Cross-axis sensitivity for those instruments can be neglected even for maximum acceleration up to about $8.0g$. Such extreme acceleration levels are only known to occur in the near-field of explosions. Maximum known acceleration for earthquakes so far did not exceed $2.5g$ [21].

Performed tests demonstrate that accelerometers used in seismological practice to record horizontal earthquake strong ground motion are sensitive to acceleration and tilt. Vertical accelerometers are only sensitive to the vertical translational acceleration. Parasitic cross-axis and angular acceleration sensitivities can practically be ignored. In the mean time, accelerograms are integrated once and twice to get velocities and displacements and those parasitic sensitivities can become additional sources of errors in strong-motion data processing.

Compared to accelerometers, seismometers are used in seismological practice to record teleseismic events with relatively low amplitudes of ground motion. Seismometers are usually characterized by relatively long natural period of about 5 s (0.2 Hz), damping ratio of ~0.6–0.7 with relatively long pendulum arm of the order of 1.0 m and longer. Testing the response of those instruments to the complex realistic input motion is beyond the scope of this study. Since pendulum arm of a seismometer is much longer than that of an accelerometer, effect of angular acceleration can become important. Residual tilt will result in a shift of baseline (similar to the effect produced to the record of an accelerometer).

Since some seismological instruments have Galperin's configuration of sensors it is also important to consider the effects of complex input ground motion including tilt on their responses, but this task is also left beyond the scope of this study.

### 3.2. Response modeling of an idealized structural system

In order to provide a realistic set of results for an inverted pendulum, its dynamic properties in terms of

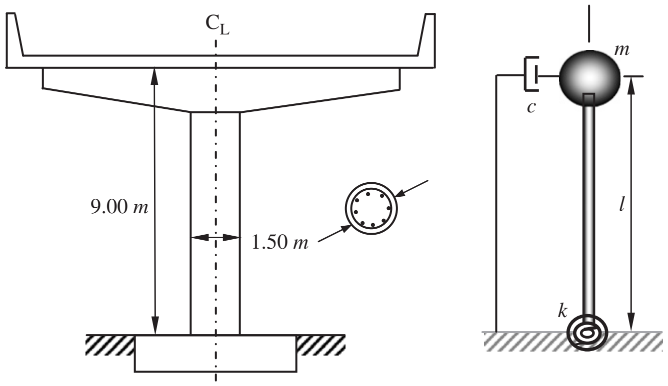


Fig. 8. Single column of a bridge bent and its idealized lumped-mass pendulum system.

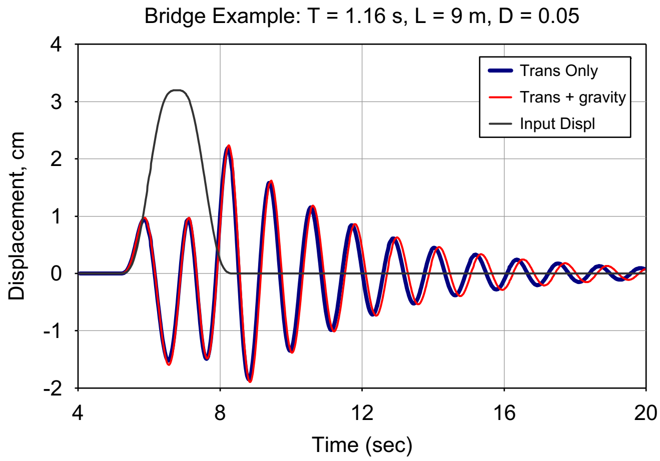


Fig. 9. Effects of gravitational acceleration on translational response.

vibration frequency, damping and height are extracted from a single-column bent design example of a highway viaduct (a part of a freeway). Such a structure is often idealized as a SDOF oscillator for response computations. This reference structure and its idealized SDOF system are demonstrated in Fig. 8. This bridge bent configuration was previously utilized as a design model by Goel and Chopra [22]. The superstructure has a total weight of 190.0 kN/m, and is supported on identical bents uniformly spaced at 39.6 m. It has a natural period of $T_n = 1.16$ s and damping ratio of $D_n = 0.05$. In the following, the idealized pendulum is analyzed under different excitation conditions including vertical, translational, angular accelerations and tilt. The projected results correspond to elastic response of the pendulum only, readers are referred to study by Kalkan and Graizer [18,23] for more details on inelastic response. In particular, impacts of the following three forcing functions and their cross-combination on the response are examined here.

1. Effect of gravity (third term on the left-hand side of Eq. (13)).
2. Effect of vertical acceleration and gravity in combination with pendulum's length and natural frequency of pendulum.
3. Effect of tilt ($\alpha$) in combination with vertical acceleration ($z''$) (second term in the right-hand side of Eq. (13)).
4. Effect of angular acceleration in combination with pendulum's length ($l_1\alpha''$) (third term in the right-hand side of Eq. (13)).

### 3.2.1. *Effect of gravitational acceleration*

Gravity factor in the left-hand side of Eq. (13) acts when pendulum is out of perfect vertical alignment. When pendulum is out of its vertical position, gravity tends to destabilize the system by producing lateral force (so called *P–Δ*, or secondary moment effect). Effect of gravity also diminishes the effective frequency of the pendulum (Eq. (13)).

Fig. 9 compares the response of an ideal pendulum sensitive to translational motion only (solution (Eq. (2)) of Eq. (1)) with the response of an inverted pendulum assuming that the input from translational motion is combined with the effect of gravity (solution (Eq. (14)) of Eq. (13)). Note that forcing function does not include tilting and vertical motion yet. Simple smoothed Dirac delta function is used as an input. Comparison of system responses demonstrates that:

- Amplitudes of displacement demand that includes gravity effect are slightly higher than that of an ideal pendulum.
- Period of free oscillation including gravity effect is slightly shifted to longer period.

The difference between the ideal response and the response that takes into account the gravity depends upon the length of the pendulum. This effect is more significant for pendulums with shorter lever arm and makes the effect of period shifting even more visible. The change in vibration period stems from eroded system stiffness with negative contribution of the geometric stiffness term (mg/l) due to gravity (see [18,23] for additional details).

### 3.2.2. *Effect of vertical acceleration*

In the next phase, horizontal and vertical acceleration and tilt from Pacoima Dam record are employed as an input excitation. First, the effect of vertical acceleration on the response is examined. Assuming that SDOF assumption holds and vertical vibration and associated axial and bending moment interaction are ignored, the influence of vertical excitation on inverted pendulum response depends upon the following two factors:

- Length of pendulum arm ($l_n$).
- Amplitude and sign of vertical ground motion ($z''$).

Fig. 10 demonstrates the effect of vertical ground motion on system relative displacement. Evidently, the polarity of the vertical motion can amplify or de-amplify the response output of a pendulum (i.e., translation response in this case). Assuming the same intensity of vertical ground motion, and depending upon its orientation and phasing,

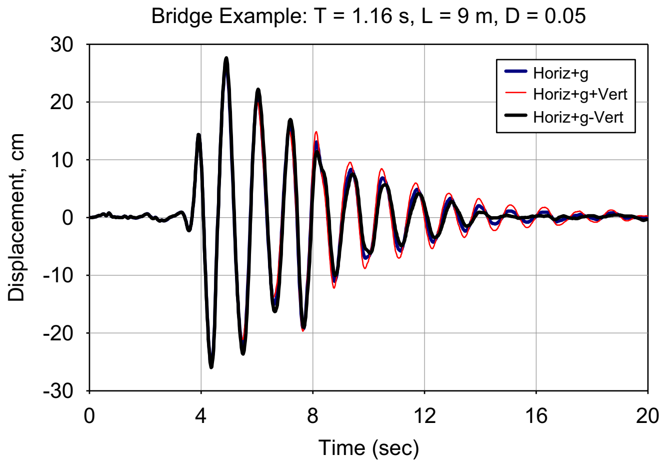


Fig. 10. Effects of polarity of vertical ground motion on translational response.

vertical motion can either increase or decrease the seismic demands. As projected in Fig. 10, peak displacement demand due to horizontal ground motion (including effect of gravity) only is lower than that of horizontal minus vertical, but almost same as horizontal plus vertical (there is still a difference at lower amplitudes between 8 and 20 s).

### 3.2.3. Effect of pendulum length

Figs. 11(top) and (bottom) manifest the response outcomes of the two pendulums with same natural period ($T_n = 1.16$ s) and damping ratio ($D_n = 0.05$), yet for different lengths (9 and 3 m). Response of a pendulum with a shorter lever arm is affected more by a combination of vertical acceleration and gravity than that with longer lever arm. Fig. 11(bottom) compares Fourier spectra of responses corresponding to pendulums with longer and shorter pendulum arms. Response of pendulum with shorter arm is influenced more by the vertical acceleration and gravity. This effect results in shifting of effective period toward longer periods by an amount of almost 10%. It happens because time dependent effective frequency replaces invariant natural frequency of pendulum ($\omega_n^2$) in left-hand side of Eq. (13).

### 3.2.4. Effect of ground tilting

Fig. 12 shows the effect of tilt and angular acceleration on displacement response of the pendulum. As Eq. (13) implies that long pendulum arm amplifies the inertial force significantly due to angular acceleration of tilting. The longer is the pendulum length, the higher is the effect of angular acceleration. Fig. 12 shows that angular acceleration of the base in case of bridge example results in up to 2.8 times increase in displacement demand. Such dramatic impacts take place particularly for systems with longer lever arm [18,23].

### 3.2.5. Effect of tilting on complete response

As can be seen from Fig. 3, $y_1 = \theta_1\, l_1$ is the displacement of the pendulum in the new position of equilibrium. Total

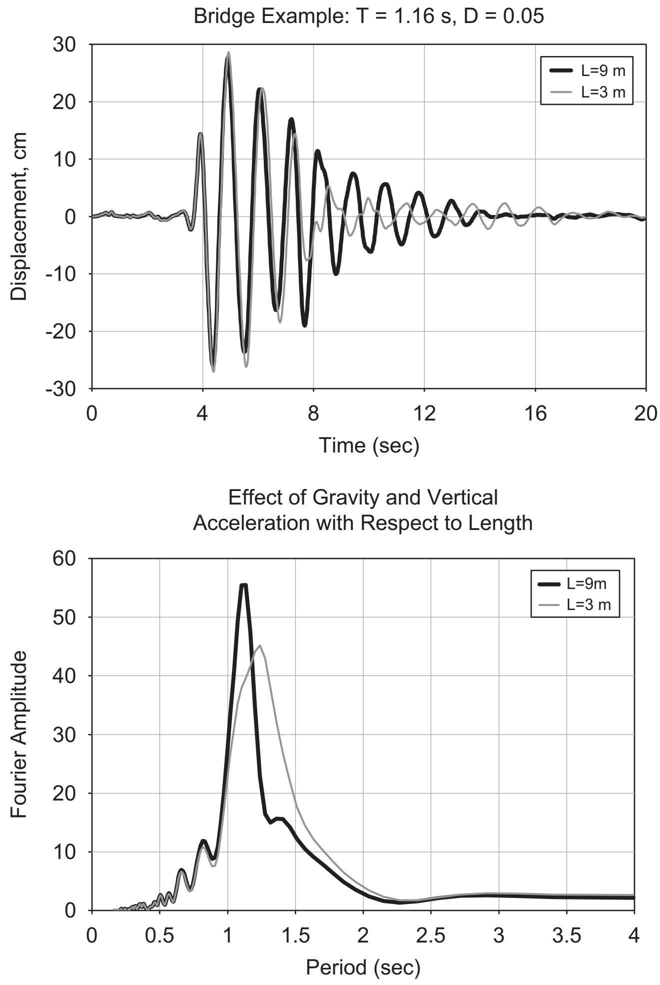


Fig. 11. Effects of pendulum length on translational response (top-panel) and Fourier amplitude spectrum (bottom-panel).

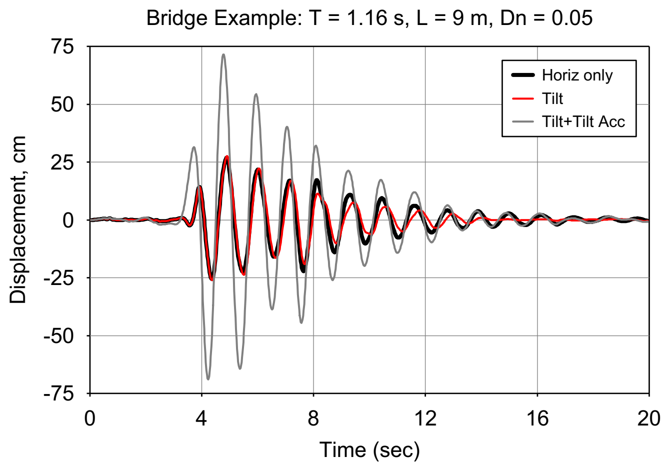


Fig. 12. Coupled effects of tilt and angular acceleration on the response.

displacement of inverted pendulum from vertical axis (for small angles $\alpha$ and $\theta$) reads:

$$Y_1 = l_1(\alpha + \theta_1) = l_1\alpha + y_1. \tag{16}$$

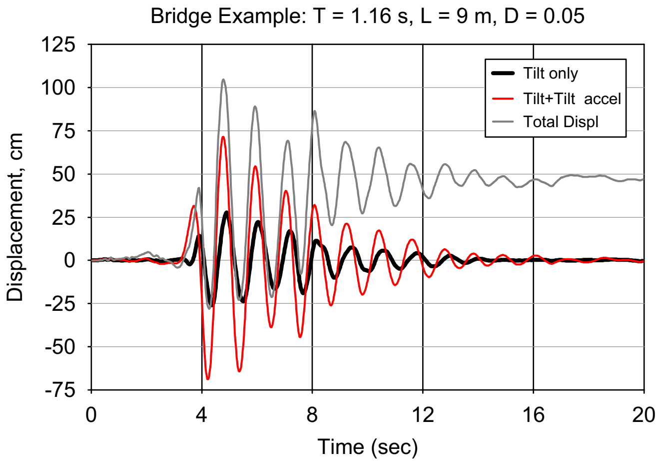


Fig. 13. Effect of tilt on translational response.

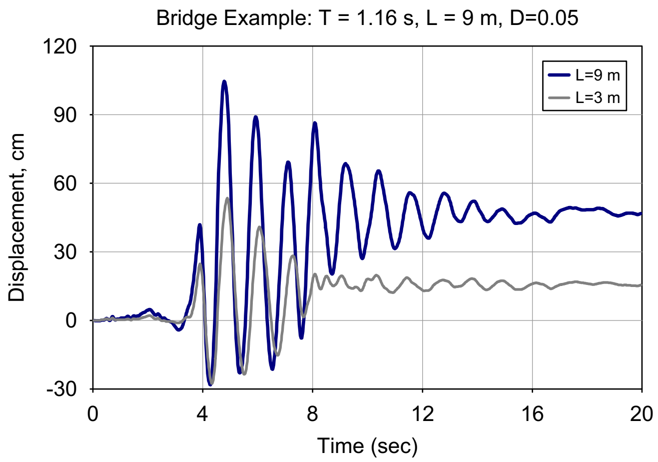


Fig. 14. Effect of pendulum length on translational response.

Fig. 13 exhibits the displacement of the pendulum at its new position of equilibrium $y_1$ and the total displacement $Y_1$ from the vertical axis. As obvious, total displacement of an inverted pendulum is noticeably amplified due to ground tilting. Finally, Fig. 14 exhibits the effect of pendulum's length on the translational response of the pendulum when the input motion includes three components of ground shaking (translational, vertical and angular accelerations and tilt). The three times difference in length almost doubles the associated displacement demand. This figure suggests that long inverted pendulums (most engineering structures) are very susceptible to tilting of the base that can result in drastic changes on the inertial force acting on mass and resultant seismic demand.

## 4. Results and conclusions

In this study, complete equations of motion for three types of pendulums are provided: mass-on-rod type, mass-on-spring type and inverted (astatic). These pendulums are used in seismological measurements and engineering practice. In contrast to classical simplified equation of the SDOF oscillator, complete equations represent comprehensive and realistic approach for computing response from seismological instruments and engineering structures by taking into account not only translational motion, but also rotations (tilt and torsion).

Parametric testing procedure is applied hereto study the response of an accelerometer and a single column of a bridge bent to various combination of forcing functions. Modeled response of a typical accelerometer has shown that for typical modern accelerometers (with natural frequency of 50 Hz and higher) used in recording strong earthquake ground motions, the effects of cross-axis sensitivity are minimal. However, as the accelerograms are integrated once and twice to get velocities and displacements, cross-axis sensitivity may serve as an additional source of errors in strong-motion data processing. Tilt of the base of an oscillator results in baseline shift and introduces long-period errors in data processing.

As compared to a common horizontal pendulum, response of an inverted pendulum is sensitive to acceleration of gravity. Gravity effect introduces non-linearity into the differential equations, and results in shift of the frequency response to lower frequencies (longer periods). This effect is higher for long-period pendulums with shorter lever arm.

Vertical acceleration (cross-axis sensitivity) affects the response of an inverted pendulum when it reaches the level close to $1.0g$. Depending upon the orientation, intense vertical acceleration may result in amplification or de-amplification of the seismic demand and of the associated response.

Inverted pendulum is sensitive to angular acceleration of tilt. This sensitivity is proportional to the length of a pendulum. Sensitivity of an inverted pendulum to angular acceleration should be taken into consideration since it can produce significant effect especially for long pendulums idealizing for instance, bridge piers, bents, elevated water tanks, telecommunication towers, etc.

Common practice of ignoring possible tilting in calculating seismic demands may result in significant underestimating of the effect produced by earthquake ground shaking. Tilting of the base of an inverted pendulum can in fact produce large inertial force and associated enhanced displacement demands to structural systems.

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